

A measure of skewness for testing departures from normality

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Abstract

We propose a new skewness test statistic for normality based on the Pearson measure of skewness. We obtain asymptotic first four moments of the null distribution for this statistic by using a computer algebra system and its normalizing transformation based on the Johnson S_U system. Finally the performance of the proposed statistic is shown by comparing the powers of several skewness test statistics against some alternative hypotheses.

Keywords: Pearson measure of skewness, normalizing transformation, powers,

1. Introduction

A test for normality is an essential problem in statistical practice. Earlier studies on tests for normality are summarized in Thode (2002). Traditionally, it is common to use skewness and kurtosis statistics $\sqrt{b_1} = m_3/m_2^{3/2}$ and $b_2 = m_4/m_2^2$, respectively, where for a random sample (X_1, X_2, \dots, X_n) , $m_r = (1/n) \sum_{i=1}^n (X_i - \bar{X})^r$, $r = 2, 3$, and 4, and $\bar{X} = (1/n) \sum_{i=1}^n X_i$. They are used for detecting skew symmetric and heavy or light tails, respectively. The Jarque–Bera test combining $\sqrt{b_1}$ and b_2 is well known as an omnibus test for normality (Jarque and Bera (1987)). The improved omnibus test was recently presented in Nakagawa et al. (2011). The above test statistics are all based on sample moments. Furthermore, the Shapiro–Wilk (W) test is famous as an extension of probability plots (Shapiro and Wilk (1965)).

In addition, the Lin–Mudholkar (LM) test is developed for asymmetric alternatives (Lin and Mudholkar (1980)). The focus of this paper is also on tests

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that detect skew symmetric. Based on $\sqrt{b_1}$ and b_2 , we propose a test statistic

$$spms = \frac{\sqrt{b_1} (b_2 + 3)}{2 (5b_2 - 6b_1 - 9)}, \quad (1)$$

and called the sample Pearson measure of skewness ($spms$). The statistic $spms$ corresponds to the Pearson measure of skewness defined by

$$pms = \frac{\sqrt{\beta_1} (\beta_2 + 3)}{2 (5\beta_2 - 6\beta_1 - 9)}, \quad (2)$$

where $\sqrt{\beta_1} = \mu_3/\mu_2^{3/2}$ and $\beta_2 = \mu_4/\mu_2^2$, and for $r = 2, 3, 4$, μ_r denotes the r th population moment about the mean. The pms is introduced by Karl Pearson considering Pearson Systems which are distinguished nine types including normal distributions (Stuart and Ord (1994)). Under Pearson Systems, the pms actually coincides with a measure of skewness

$$\frac{\text{mean} - \text{mode}}{\sigma},$$

where σ is the standard deviation. As the mean and the mode coincide in a symmetric population, the distance from the mean to the mode can be treated as a measure of skewness.

The first four moments of the null distribution for $spms$ are explained in Section 2. Based on the Johnson S_U system, we obtain a transformation to approximate the normality of the null distribution of $spms$ in Section 3. In Section 4, the performance of $spms$ is shown by comparing the powers of several skewness test statistics against six asymmetric alternatives.

2. Approximate moments of the null distribution for $spms$

In this section, we assume that a sample (X_1, X_2, \dots, X_n) is drawn from a normal population. The $spms$ is invariant under the origin, and the scale changes because of the invariance of $\sqrt{b_1}$ and b_2 . It is clear that $spms$ is symmetric. Figure 1 shows a histogram of $spms$ with $n = 100$.

To find the approximate moments of the distribution for $spms$, a large amount of symbolic computation is required. Niki and Nakagawa (1995) described a method for this purpose as follows:

- (1) Set $U = \sqrt{n} (m_2 - 1)$, $V = \sqrt{n} m_3$, $W = \sqrt{n} (m_4 - 3)$, and expand $spms$ as power series in terms of $1/\sqrt{n}$:

$$\begin{aligned} spms = & \frac{1}{\sqrt{n}} \left(\frac{1}{2} V \right) + \frac{1}{n} \left(\frac{5}{4} U V - \frac{1}{3} V W \right) + \frac{1}{n\sqrt{n}} \left(\frac{79}{16} U^2 V \right. \\ & - \frac{13}{6} U V W + \frac{1}{2} V^3 + \frac{5}{18} V W^2 \Big) + \frac{1}{n^2} \left(\frac{517}{32} U^3 V - \frac{263}{24} U^2 V W \right. \\ & \left. + \frac{9}{4} U V^3 + \frac{95}{36} U V W^2 - \frac{3}{4} V^3 W - \frac{25}{108} V W^3 \right) + O(n^{-5/2}). \end{aligned}$$

We remark that $U = O(1)$, $V = O(1)$, and $W = O(1)$.

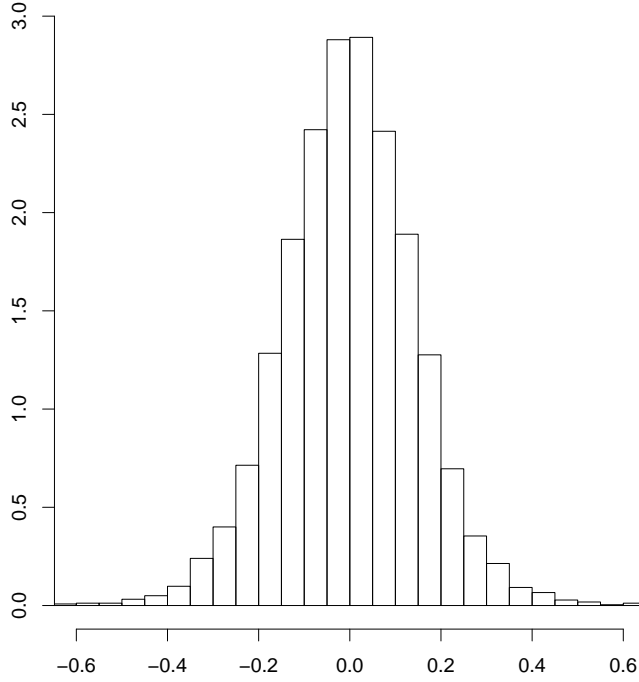


Figure 1: Histogram of $spms$ with $n = 100$ (10^4 replications)

- (2) Take expectations term by term. Thus, we obtain the first approximate moment of $spms$.
- (3) The second, third, and fourth moments are obtained from $spms^2$, $spms^3$, and $spms^4$, respectively.

In step (2), each $U^i V^j W^k$ ($0 \leq i + j + k \leq 8$) is a symmetric polynomial of X_1, X_2, \dots, X_n . We emphasize that the algorithm for the change of bases of symmetric polynomials described in Niki and Nakagawa (1995) is applied in computing expectations $E[U^i V^j W^k]$. The algorithm has been implemented in REDUCE (Hearn (2004)) with utility programs.

Let λ_r be the r th ($r = 2, 4$) moment of the distribution for $spms$ drawn from a normal population, each having the following form:

$$\lambda_2 = \left(\frac{3}{2}\right) n^{-1} + 41n^{-2} + \left(\frac{6511}{2}\right) n^{-3} + O(n^{-4}), \quad (3)$$

$$\lambda_4 = \left(\frac{27}{4}\right)n^{-2} + 414n^{-3} + O(n^{-4}). \quad (4)$$

Note that all odd moments are zero. From (3), $spms$ is asymptotically normally distributed with mean 0 and variance $3/(2n)$. Thus, we obtain

$$\beta_2(spms) = \frac{\lambda_4}{\lambda_2^2} = 3 + 20n^{-1} + \left(\frac{48544}{3}\right)n^{-2} + \left(\frac{10386704}{9}\right)n^{-3} + O(n^{-4}), \quad (5)$$

which will be used for obtaining a normalizing transformation.

3. Transformation

A normal approximation of the null distribution for $spms$ is obtained using a transformation described in Johnson (1949), given as follows:

Theorem 1. *Assume that the null hypothesis of normality is true and let*

$$Y = \frac{spms}{\sqrt{\lambda_2}}, \quad (6)$$

$$W^2 = -1 + \sqrt{2(\beta_2(spms) - 1)}, \quad (7)$$

$$\delta = \frac{1}{\sqrt{\log W}}, \quad (8)$$

$$\alpha = \sqrt{\frac{1}{W^2 - 1}}. \quad (9)$$

Then

$$Z = \delta \log \left(\frac{Y}{\alpha} + \sqrt{1 + \left(\frac{Y}{\alpha}\right)^2} \right) \quad (10)$$

is approximately a standard normal variate with mean zero and variance unity.

This transformation is an analogy of the normalizing transformation of the null distribution for $\sqrt{b_1}$ given by D'Agostino (1970).

Figure 2 shows a histogram of Z with $n = 200$ when 10^6 random normal samples are generated. The dashed line denotes the probability density function of the standard normal distribution. Both one-sided and two-sided tests at any desired levels of significance can be performed. For example, for a two-sided test with a 0.05 level of significance, the null hypothesis is rejected if $|Z| > 1.96$.

Monte Carlo simulations are performed for judging the accuracy of the transformation. For samples of sizes $n = 100, 150, 200, 300, 500$, and 1000 , 10^6 random normal samples are generated. For each sample, the $spms$ statistic is computed and classified into the intervals formed by the two-sided tests with levels of significance 0.01, 0.05, 0.10, and 0.20. The entries in Table 1 show the Monte Carlo relative frequencies for the intervals. For $n \geq 500$, the Monte Carlo results agree to at least two decimal places for any significance level. Even if $n = 100$, the results are good for 0.01 level of significance.

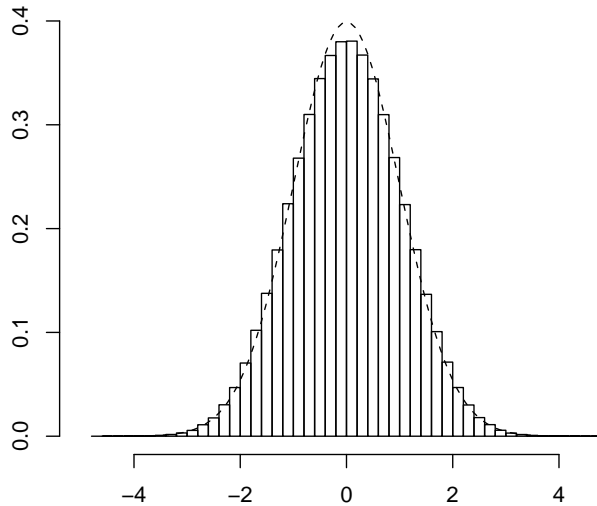


Figure 2: Histogram of $Z(10)$ with $n = 200$ (10^6 replications) and the standard normal probability density function (dashed curve)

4. Power study

This section demonstrates a comparison of the powers of *spms* with $\sqrt{b_1}$, W , and LM tests. Monte Carlo simulations are conducted with samples of sizes $n = 40, 50, 60, 80$, and 100 and 10^4 replications. Two-sided tests with a 5% significance level are carried out against six alternative distributions: beta distributions with two parameters $(p, q) = (2, 1), (3, 2)$; gamma distributions with shape parameters $\alpha = 2, 3$; Weibull distributions with shape parameters $\alpha = 2$; and log-normal distribution with mean $\mu = 0$ and standard deviation $\sigma = 1/2$. The first two distributions have light tails and the rest have heavy tails.

The powers of the above four statistics are summarized in Table 2 and Figures 3–8 (*spms*(\circ), $\sqrt{b_1}$ (\triangle), W ($+$), and LM (\times)). In most cases, the power of the *spms* test is better than that of the $\sqrt{b_1}$ test. The power of the *spms* test is superior to that of all other tests in the first two cases. The *spms* test is comparable when the alternative distributions have heavy tails.

Table 1: Monte Carlo probabilities for various levels of significance using $Z(10)$ with 10^6 replications per sample size

Presumed levels	sample size					
two-sided test	100	150	200	300	500	1000
0.01	0.0112	0.0099	0.0098	0.0098	0.0102	0.0100
0.05	0.0595	0.0537	0.0519	0.0513	0.0504	0.0501
0.10	0.1207	0.1097	0.1057	0.1027	0.1009	0.1003
0.20	0.2368	0.2191	0.2113	0.2052	0.2019	0.2004

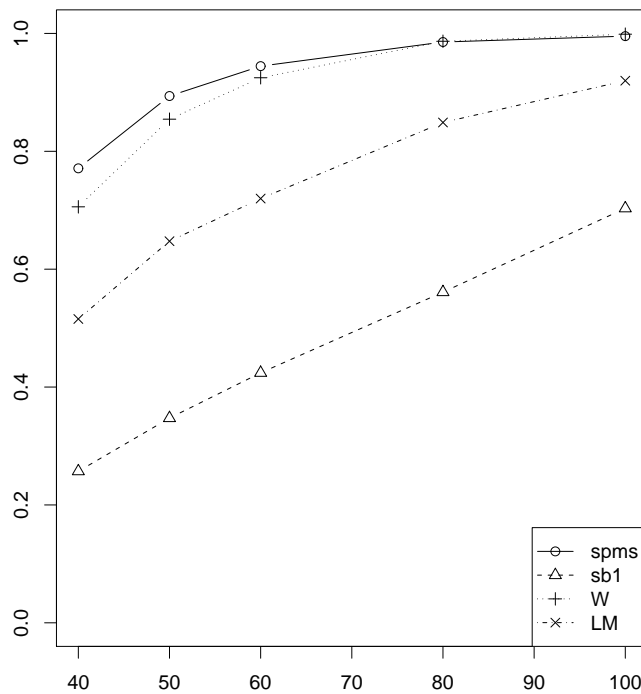


Figure 3: Powers of the tests when the alternative distribution is Beta ($\alpha = 2, \beta = 1$)

Table 2: Monte Carlo powers of some tests for normality, level of significance 0.05, population skewness $\sqrt{\beta_1}$, and kurtosis β_2

Alternative hypotheses	$\sqrt{\beta_1}$	β_2	n	$spms$	$\sqrt{b_1}$	SW	LM
Beta ($\alpha = 2, \beta = 1$)	-0.57	2.4	40	0.771	0.257	0.706	0.515
			50	0.894	0.347	0.855	0.647
			60	0.945	0.424	0.925	0.720
			80	0.985	0.561	0.987	0.849
			100	0.995	0.703	0.999	0.920
Beta ($\alpha = 3, \beta = 2$)	-0.29	2.4	40	0.284	0.048	0.153	0.095
			50	0.362	0.059	0.213	0.141
			60	0.406	0.062	0.259	0.153
			80	0.510	0.081	0.400	0.203
			100	0.600	0.114	0.518	0.263
Wiebull ($\alpha = 2$)	0.63	3.3	40	0.297	0.313	0.322	0.369
			50	0.430	0.377	0.420	0.469
			60	0.556	0.432	0.500	0.541
			80	0.728	0.570	0.675	0.673
			100	0.847	0.687	0.789	0.798
Gamma ($\alpha = 3$)	1.16	5.0	40	0.611	0.684	0.708	0.751
			50	0.803	0.778	0.829	0.849
			60	0.900	0.851	0.894	0.912
			80	0.976	0.938	0.968	0.968
			100	0.994	0.978	0.991	0.991
Gamma ($\alpha = 2$)	1.41	6.0	40	0.810	0.823	0.880	0.893
			50	0.935	0.895	0.955	0.950
			60	0.980	0.937	0.981	0.979
			80	0.998	0.988	0.998	0.996
			100	1.000	0.998	1.000	1.000
log normal ($\mu = 0, \sigma = 1/2$)	1.80	8.9	40	0.764	0.833	0.846	0.884
			50	0.909	0.908	0.932	0.940
			60	0.963	0.945	0.962	0.975
			80	0.993	0.986	0.992	0.993
			100	0.999	0.996	0.999	0.999

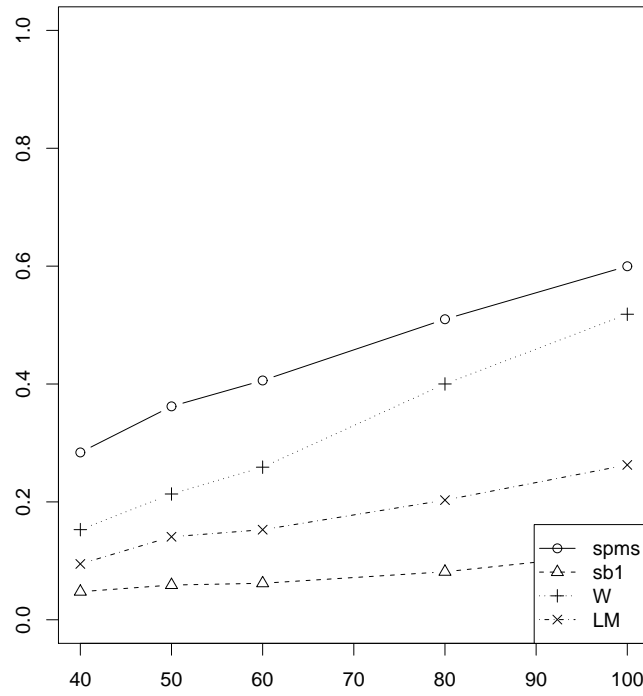


Figure 4: Powers of the tests when the alternative distribution is Beta ($\alpha = 3, \beta = 2$)

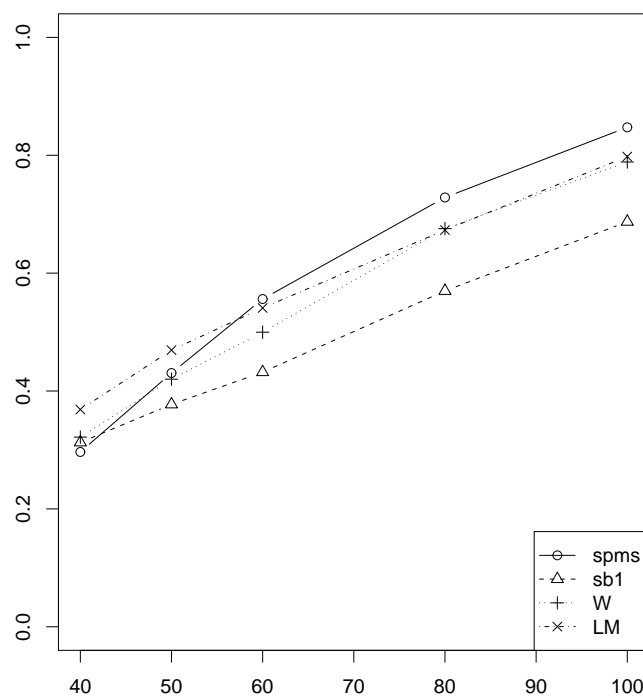


Figure 5: Powers of the tests when the alternative distribution is Weibull ($\alpha = 2$)

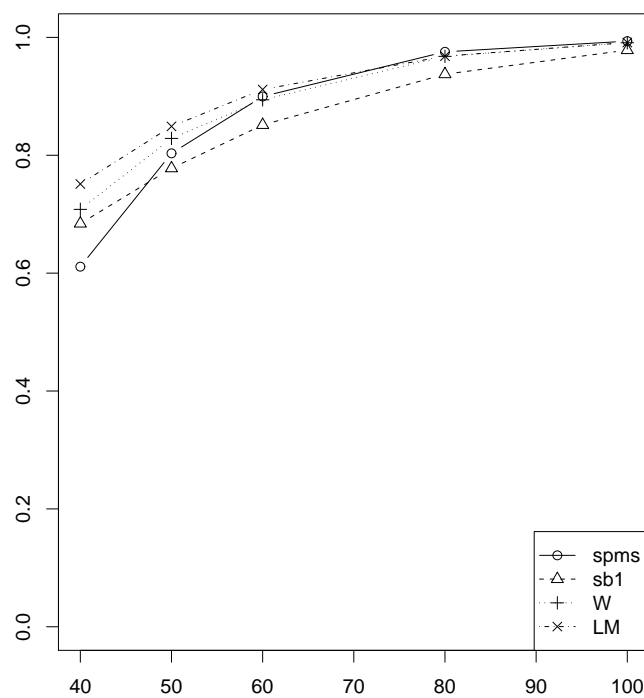


Figure 6: Powers of the tests when the alternative distribution is Gamma ($\alpha = 3$)

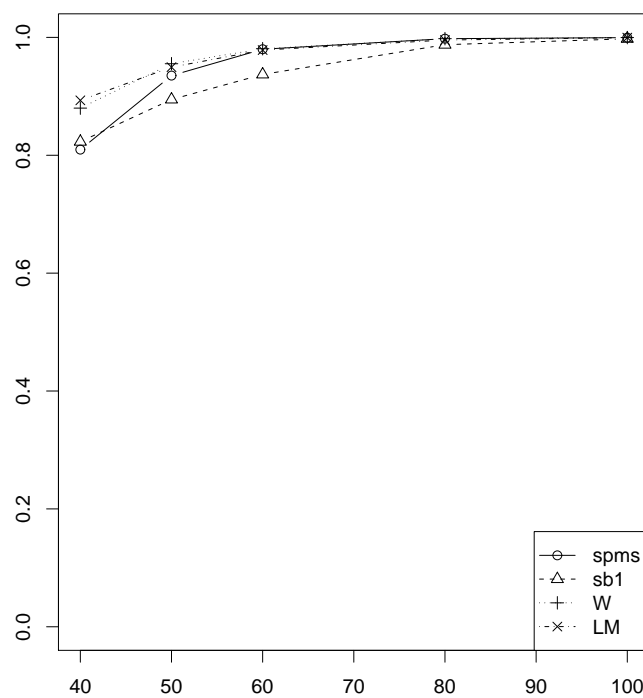


Figure 7: Powers of the tests when the alternative distribution is Gamma ($\alpha = 2$)

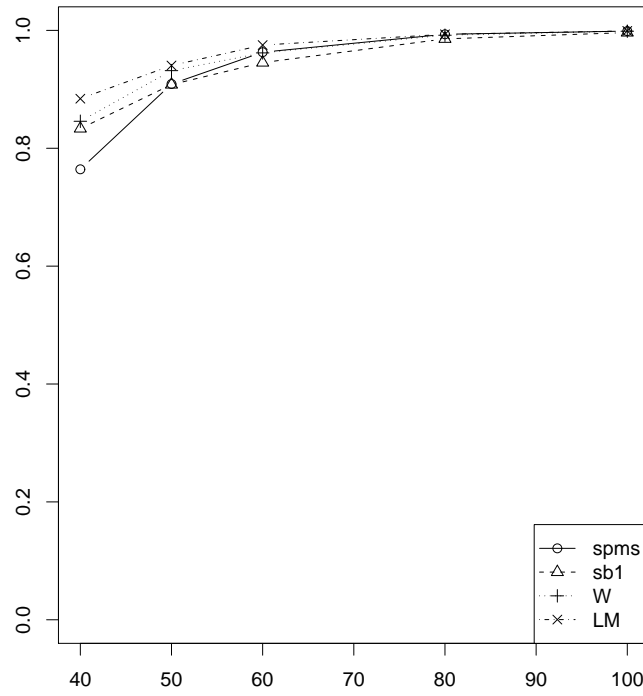


Figure 8: Powers of the tests when the alternative distribution is log normal ($\mu = 0, \sigma = 1/2$)

5. Concluding remarks

Corresponding to the Pearson measure of skewness pms (2), we propose $spms$ (1) as a skewness test statistic for normality. We obtain the normalizing transformation (10) of the null distribution for $spms$ based on the second and fourth approximate moments shown in (3) and (4). For a moderate sample size, such as $n \geq 200$, the transformation is valid in practice, as shown in Table 1.

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